# Asymmetrical Bending and Vibration of a Conical Shell Finite Element

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## Theme

STIFFNESS and mass matrices are formulated for a trapezoidal conical shell finite element by an integration technique in connection with the method of synthetic division. The element is shown to be efficient by examples of asymmetrical bending and free vibration of conical shells.

## Content

Conical shells constitute an important structural component in the aerospace vehicles. Asymmetrical bending and vibration are problems of considerable interest. A conical shell finite element with trapezoidal shape seems to be more advantageous than the closed-ring shape when asymmetrical loadings, longitudinal stiffeners, cut-outs, etc. are considered. Such an element is shown in Fig. 1. The element has the same 48 degrees of freedom as those assumed in Ref. 1. The edges 1–3 and 2–4 are two generators of the shell.

The element is described by the following definitions of geometrical parameters:

$$d\eta = B\,d\theta; \qquad B = R_o + \xi \sin\phi$$
$$R_\xi = \infty; \qquad R_\eta(\xi) = (s+\xi)\tan\phi \tag{1}$$

The Lamé parameter $B$ and the circumferential radius of curvature $R_\eta$ vary linearly with coordinate axis $\xi$.

Each of the $u$, $v$, and $w$ displacements is defined as the product of 16 shape functions $f_i$ and 16 degrees of freedom $q_i$. The shape functions are the products of nondimensionalized first-order Hermitian polynomials $h$

$$f_i(\xi,\eta) = a^J \Delta\theta^K h_M(\alpha) h_N(\beta) \qquad (i = 1, 2, \ldots, 16) \tag{2}$$

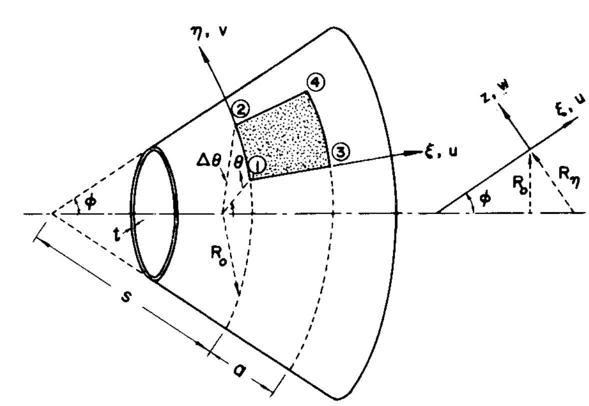


**Fig. 1 The trapezoidal conical shell finite element.**

where $\alpha = \xi/a$; $\beta = \theta/\Delta\theta$; the integers $J$, $K$, $M$, and $N$ are related in a pattern which is given in the full paper. Following Eq. (2), the Lamé parameter becomes

$$B = a\sin\phi(\alpha+\delta) \quad \text{with} \quad \delta = s/a \tag{3}$$

The strain-displacement relations are

$$\varepsilon_\xi = u_\xi$$
$$\varepsilon_\eta = (1/B)v_\theta + (\sin\phi/B)u + (\cos\phi/B)w$$
$$\varepsilon_{\xi\eta} = v_\xi + (1/B)u_\theta - (\sin\phi/B)v$$
$$\kappa_\xi = -w_{\xi\xi}$$
$$\kappa_\eta = (\cos\phi/B^2)v_\theta - (1/B^2)w_{\theta\theta} - (\sin\phi/B)w_\xi$$
$$\kappa_{\xi\eta} = (\cos\phi/B)v_\xi - (2\sin\phi/B^2)v - (2/B)w_{\xi\theta} + (2\sin\phi/B^2)w_\theta \tag{4}$$

Based on Eqs. (2–4) and the standard energy expressions for shells, the equations of motion can be obtained by applying Lagrange's equation. The mass and stiffness matrices are obtained as

$$\underset{48\times 48}{[m]} = \rho t \int_{\text{area}} \begin{bmatrix} ff & 0 & 0 \\ 0 & ff & 0 \\ 0 & 0 & ff \end{bmatrix} d\xi\, B\, d\theta \tag{5}$$

$$\underset{48\times 48}{[k]} = \frac{Et}{B(1-v^2)} \int_{\text{area}} \begin{bmatrix} A_{uu} & A_{uv} & A_{uw} \\ A_{vu} & A_{vv} & A_{vw} \\ A_{wu} & A_{wv} & A_{ww} \end{bmatrix} d\xi\, B\, d\theta \tag{6}$$

in which

$$[A_{uu}] = Bf_\xi f_\xi + \mu^2 rff + 2\mu v f_\xi f + (1-v) r f_\theta f_\theta/2$$
$$[A_{vu}] = \mu r f_\theta f + v f_\theta f_\xi + (1-v) f_\xi f_\theta/2 - (1-v)\mu r f f_\theta/2$$
$$[A_{vv}] = r f_\theta f_\theta + (Bf_\xi f_\xi - \mu f_\xi f - \mu f f_\xi + \mu^2 rff)(1-v)/2 + \gamma\lambda^2 r^3 f_\theta f_\theta + 2\gamma\lambda^2 r(1-v)(f_\xi f_\xi - 2\mu r f_\xi f - 2\mu r f f_\xi + 4\mu^2 r^2 ff)$$
$$[A_{wu}] = \lambda\mu r ff + v\lambda f f_\xi$$
$$[A_{wv}] = \lambda r f f_\theta - \gamma\lambda r(r^2 f_{\theta\theta} f_\theta + \mu r f_\xi f_\theta + v f_{\xi\xi} f_\theta) + 4\gamma\lambda r(1-v)(-f_{\xi\theta} f_\xi + \mu r f_\theta f_\xi + 2\mu r f_{\xi\theta} f - 2\mu^2 r^2 f_\theta f)$$
$$[A_{ww}] = \lambda^2 rff + \gamma(Bf_{\xi\xi} f_{\xi\xi} + r^3 f_{\theta\theta} f_{\theta\theta} + \mu r^2 f_{\theta\theta} f_\xi + \mu r^2 f_\xi f_{\theta\theta} + \mu^2 r f_\xi f_\xi) + \gamma v(r f_{\xi\xi} f_{\theta\theta} + r f_{\theta\theta} f_{\xi\xi} + \mu f_{\xi\xi} f_\xi + \mu f_\xi f_{\xi\xi}) + 8\gamma r(1-v)(f_{\xi\theta} f_{\xi\theta} - \mu r f_{\xi\theta} f_\theta - \mu r f_\theta f_{\xi\theta} + \mu^2 r^2 f_\theta f_\theta) \tag{7}$$

where $\lambda = \cos\phi$; $\mu = \sin\phi$; $\gamma = t^2/12$; $r = 1/B$; $v$ = Poisson's ratio; and the term $ff$ denotes a $16\times 16$ matrix $\underset{16\times 1}{\{f\}}\underset{1\times 16}{\{f\}^T}$.

Equations (2, 3, 5–7) reveal that five patterns of integration are involved

$$C_1(i,j) = \int_0^1 h_M(\alpha)h_N(\alpha)\,d\alpha; \quad C_2(i,j) = \int_0^1 (\alpha+\delta)h_M(\alpha)h_N(\alpha)\,d\alpha;$$
$$C_3(i,j) = \int_0^1 \frac{1}{(\alpha+\delta)} h_M(\alpha)h_N(\alpha)\,d\alpha;$$
$$C_4(i,j) = \int_0^1 \frac{1}{(\alpha+\delta)^2} h_M(\alpha)h_N(\alpha)\,d\alpha; \tag{8}$$
$$C_5(i,j) = \int_0^1 \frac{1}{(\alpha+\delta)^3} h_M(\alpha)h_N(\alpha)\,d\alpha$$

where the products of $h_M(\alpha)$ and $h_N(\alpha)$ are sixth-order polynomials of $\alpha$. The division of functions $h_M(\alpha)h_N(\alpha)$ by binomials

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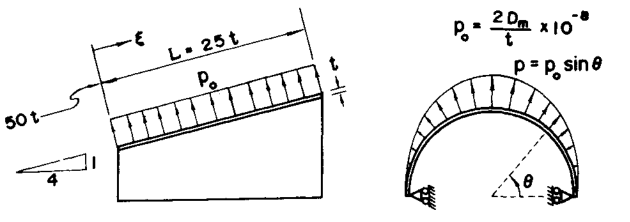


**Fig. 2 The geometry and loading for one half of a conical shell.**

$(\alpha+\delta)^n$ is performed by the method of synthetic division. The quotients are polynomials of $\alpha$ with order ranging from $\alpha^{6-n}$ to $(\alpha+\delta)^{-n}$. By integrating these quotients from 0 to 1, the constant $C$-matrices, defined in Eq. (8), can be obtained.

The application of the method of synthetic division is demonstrated by finding $C_5(1,4)$ with the assumption that $a = 10$ in., $s = 20$ in., and $\delta = 2$

$$C_5(1,4) = \int_0^1 \left[ 2\alpha^3 - 17\alpha^2 + 81\alpha - 297 + \frac{945}{\alpha+2} - \frac{864}{(\alpha+2)^2} + \frac{324}{(\alpha+2)^3} \right] d\alpha = \frac{2}{4} - \frac{17}{3} + \frac{81}{2} - 297 + 945 \ln\left(1 + \frac{1}{2}\right) - \frac{864}{2(1+2)} + 324 \frac{(1+4)}{2^2(1+2)^2} \quad (9)$$

With the availability of constant matrices $C$, the mass and stiffness matrices can be obtained. It is noted that the variable $\delta$ varies with different elements. Since $\delta$ is absent in $C_1(i,j)$, $C_1$-matrix is constant for every element. This is the case in the element where the Lamé parameter is constant.[1,2] This matrix has been presented in Ref. 2.

For formulating the mass and stiffness matrices for a single element, it requires about 7 sec central processing time for CDC 6500. Had it not been possible to organize the integration method so as to fit computer programing, it would have required one to perform thousands of lengthy separate integration. In that case, the compiling of the explicitly formulated matrices for one element would require approximately 20 sec central processing time.

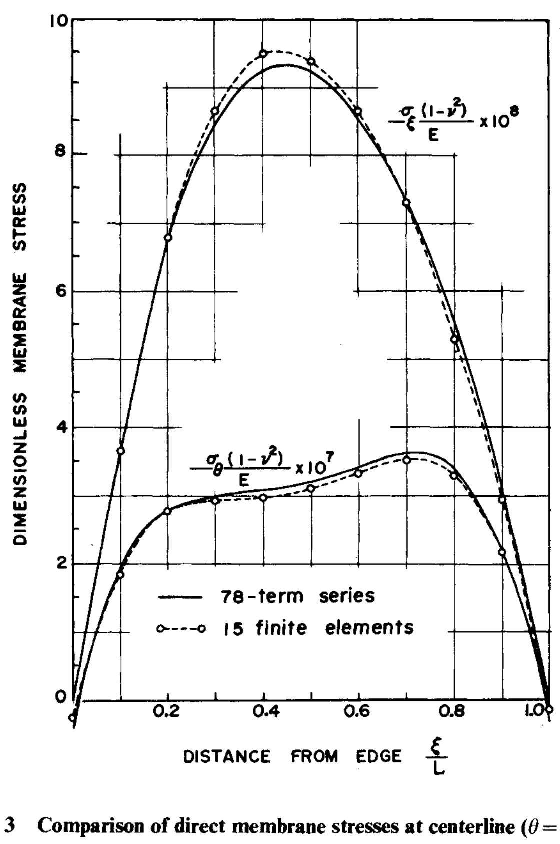


**Fig. 3 Comparison of direct membrane stresses at centerline ($\theta = \pi/2$).**

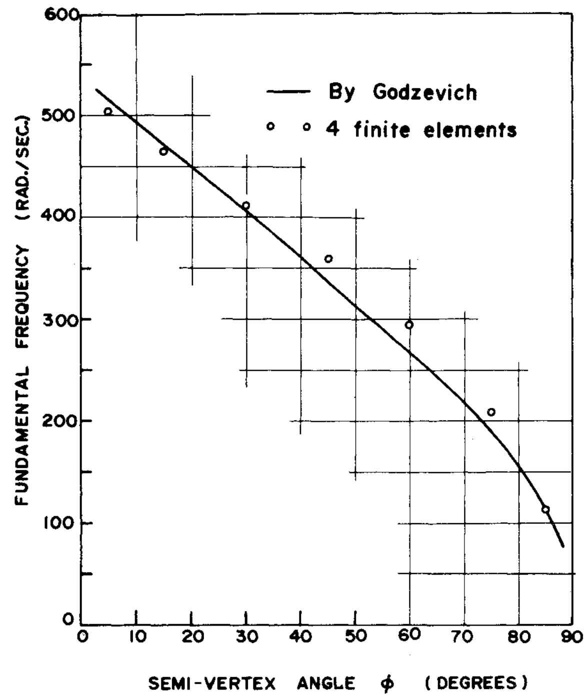


**Fig. 4 Comparison of the lowest frequencies of the conical shell with various semivertex angles (for 1/2 meridional wave).**

1) *Static analysis*: Fig. 2 shows one half of a truncated conical shell with four edges simply supported. The loading is also shown in Fig. 2. The details of the boundary conditions are given in the full paper. The results for the direct membrane stresses at centerline ($\theta = \pi/2$) obtained by using a $3 \times 5$ mesh for idealizing a quarter of the shell are shown in Fig. 3. The results are in good agreement with a 78-term series solution.[3] The proof of convergence of deflection profiles by using $2 \times 2$, $3 \times 3$, $3 \times 4$, and $3 \times 5$ meshes is given in the full paper. The satisfactory results of membrane stresses and bending moments are also given in the full paper.

2) *Free vibration*: a truncated conical shell with $E = 10^7$ psi, $\nu = 0.3$, $\rho = 0.26 \times 10^{-3}$ slug/in.$^3$, $t = 0.04$ in., $R_o = 40$ in., and $L = 40$ in. is studied. The simply supported edge conditions, which were treated before[4] are considered. A segment bounded by $\xi = 0$, $\xi = L$, $\theta = 0$, and $\theta = \pi/2n$ is under analysis and a $2 \times 2$ mesh is used. By assuming 1/2 meridional wave, the lowest frequencies and the corresponding modes were found for various semi-vertex angles. The results are shown in Fig. 4. Corresponding to $\phi = 5°, 15°, \ldots, 85°$, the circumferential wave numbers are 8, 9, 10, 11, 11, 10, and 7, respectively. The results are in satisfactory agreement with those obtained by Galerkin's method.[4] The results of a convergence study of frequencies with mesh refinement are given in the full paper.

## References


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